PFS 29-6 1/25/16 ________________

Prime Numbers

Choose one project that's an appropriate challenge for you, and do a good job with it! If you have time after that, you're welcome to work on other parts of the POW. Try to work with other students, family members, friends and teachers.

We can write any word in English by combining some of the 26 letters of the English alphabet. And in science, all compounds are made up of some of the 92 elements. In math, prime numbers play a similar role, as every whole number can be built up from prime numbers.

Prime numbers are numbers with exactly two factors. Here's a way to picture this. If we have 13 square tiles, the only rectangle we can make from them is a "snake," 1 x 13. But if we have 12 tiles, we can make 1 x 12, 2 x 6, and 3 x 4. Thirteen is a prime number, and its factors are 1 and 13. Twelve is a composite number, and its factors are 1, 2, 3, 4, 6, and 12. The number one is a "unit," and all greater numbers are prime or composite.

People have been studying primes for over two thousand years. We've learned some things about them, but there are still many mysteries!

1) Make a neat chart of the first 20 (or more) counting numbers. Label each number as a unit, a prime, or a composite. For the composites, write their factors. Your chart should look something like this:

1	Unit	
2	Prime	
3	Prime	
4	Composite	1 x 4, 2 x 2
5	Prime	
6	Composite	1 x 6, 2 x 3

After you complete your chart, write down anything you notice. Do the prime numbers seem to show up in a regular pattern?

2) In 1742 Christian Goldbach wrote a letter to the great mathematician Leonhard Euler, wondering if it was true that every even number greater than 2 could be written as the sum of two prime numbers. This is an unsolved problem! No one has ever found an even number that couldn't be written this way, but no one has found a proof that it will always work. Do some work on Goldbach's Conjecture. Make a chart starting like this:

$4 = 2 + 2$
$6 = 3 + 3$
$8 = 3 + 5$
$10 = 3 + 7 = 5 + 5$

Write down anything you notice!

3) Here's another way to find prime numbers, called The Sieve of Eratosthenes. Write as many counting numbers as you can on graph paper using six columns. Your chart will start like this:

1	2	3	4	5	6
7	8	9	10	11	12

Put a triangle around 1 -- it's a unit. Put a circle around 2 -- it's the first prime. Now cross out all the multiples of 2, since they're all composite. Put a circle around 3 -- it's the next prime. Now cross out all the multiples of 3. Circle the next number, 5 -- it's the next prime! Etc. It's automatic! The next prime number will always be the next number that's not crossed out! Find out when and where Eratosthenes lived, and what else he accomplished!

4) Every number has its own fingerprint, or formula -- the set of prime numbers that multiply up to it. For example, 30 = 2 x 3 x 5. Learn how to use factor trees to find out the prime factorization of any number. Make a chart showing prime factorizations. Your chart will start like this:

1 = 1	7 = 7
2 = 2	8 = 2 x 2 x 2
3 = 3	9 = 3 x 3
4 = 2 x 2	10 = 2 x 5
5 = 5	11 = 11
6 = 2 x 3	12 = 2 x 2 x 3

5) How many prime numbers are there? At first this must have seemed like an unsolvable problem. They do get rarer as we go higher and higher, but can we go to infinity to see if there are always more of them? The great mathematician Euclid, who lived in Alexandria, Egypt, around 300 B.C.E., wrote a very important book called *The Elements.* In this book he gave a wonderful proof that the set of primes is infinite! Try to learn his proof, and explain it in your own words.

6) How can we decide if a certain number is prime? For example, is 91 prime? How did you decide? Can you check which of these numbers are prime:

137 139 141 143 221 223 1,001

Some modern codes depend on the fact that it's very hard to factor large numbers. One such code is called RSA. Learn more about this!

6) Sophie Germain primes are named after a French mathematician who lived about 200 years ago. If you double a prime and add one and you get another prime, the first prime is a Sophie Germain prime. (For example, 5 x 2 = 10, and 10 + 1 = 11, so 5 is a Sophie Germain prime.) Make a list of them. What do you notice? Germain used her special type of primes numbers to create a partial proof of Fermat's Last Theorem.

7) Speaking of Fermat, his “little theorem” says that

a^p - a is always divisible by p, if p is a prime number.

Test this! For example, is $2^5 - 2$ divisible by 5? Try other examples! Is $4^3 - 4$ on the 3 times table? Is $3^{11} - 3$ on the 11 times table?

7) Some primes can be written as the sum of two squares, some can't. (For example, 5 = 1 + 4; but 7 is not the sum of two square numbers.) Explore this! Find a rule!

8) The gaps between primes get bigger and bigger (on average) as we look at higher and higher numbers. But how big can that gap get? There’s actually a proof that there are gaps between primes numbers as large as any finite number you can name! The proof uses factorials, and maybe you can learn it!

8.5) Speaking of factorials, there’s Wilson’s Theorem, which has a long interesting history. It says that (n – 1)! + 1 will be divisible by n if and only if n is prime. Let’s try it for seven. Seven minus one is six, and 6! = 720. Is 721 on the seven times table? Yup! Is seven prime? Yup! Try it with other numbers! Learn more about Wilson’s Theorem. Why isn’t Wilson’s Theorem a useful way to decide if a big number is prime?

8.75) Late-breaking news! There’s also “Richard’s Theorem,” a little variation on Wilson’s Theorem, that has an easier proof. Ask us about it!

9) Even though the gaps can be as big as you want, it looks like there are still twin primes as we get higher and higher. (Twin primes are primes that are two apart, such as 17 and 19, or 29 and 31. Can you find more twin primes? Mathematicians think that there are an infinite number of twin primes, although that hasn’t been proven yet. But recent work by Yitang Zhang, James Maynard and Terence Tao has gotten us very close to a proof. Learn more about twin primes!

10) A perfect number is the sum of its factors (other than itself). For example, 6 is a perfect number because 6 = 1 + 2 + 3. Euclid gave a formula for finding even perfect numbers by multiplying a power of two (2, 4, 8, 16...) by a number one less than the next power of two. For example, 2 x 3 = 6. Here’s another example: 4 x 7 = 28. Check to see if 28 is perfect! 8 x 15 = 120. Is 120 perfect? If not why not? Some mathematicians think prime numbers were first discovered and named in this search for perfect numbers. Numbers one less than a power of two that are prime are called Mersenne primes. Learn about GIMPS -- the Great Internet Mersenne Prime Search. What is the highest known prime number? How did Euler add to Euclid’s proof many centuries later? What happens if you write perfect numbers in Base Two? (And by the way, no one knows if there are any odd perfect numbers. That’s an unsolved problem!)

11) Have fun!