Stories about Infinity

1) Learn to tell Infinity Hotel, Part 1. Tell it to a family member. Make a note of how your performance went and how the audience reacted. Then do the same with Part 2.

2) Are there more counting numbers than square numbers? Hint: pair them up!

3) Zeno of Elea was a Greek philosopher who lived about 2500 years ago. He made up a series of *paradoxes.* A paradox is a puzzling story that seems true, but can't be true! For example, he said you can't cross a room because first you half to cross half of the room. But before you reach the halfway point, you have to reach half of that—a quarter of the way across the room. But before you do that… And uh oh! You have an infinite number of things to do before you can even get halfway. What do you think?

4) When I was a kid my brother would teach me math he was learning in school. (He's six years older than me.) And sometimes he'd try to stump me with a puzzle or problem. I remember when he told me Zeno's Paradox of Achilles and the Tortoise. It drove me crazy! I said, "Steve! This doesn't make sense! What's the answer?" But he said, "I don't understand it either!"

Here's one way to tell the story. Achilles can run ten times faster than the tortoise, so he gives the tortoise a 100 m head start. Then, Zeno says, Achilles can never catch up with the tortoise or pass him! Zeno keeps asking, where will the tortoise be when Achilles reaches the place the tortoise just was? For example, when Achilles reaches the 100 m mark, where the tortoise started, the tortoise will be at the 110 m mark. You might want to make a chart showing how far each runner is from the starting line at certain times. Start like this and keep going:

Achilles	**Tortoise**
0 m	100 m
100 m	110 m
110 m	111 m

Can you make a story out of this? Can you resolve the paradox? When I was nine or ten, I knew Achilles would win, but I couldn't explain what was wrong with Zeno's argument. If Achilles can catch up with the tortoise, how far from the starting line will they be when that happens?

5) How many primes are there? Is there an infinite number of them? Imagine two people arguing about this, say 2,500 years ago:

> Joe: Well, we all know the primes get rarer and rarer as we go up through the numbers. Eventually there are no more. With numbers so big, they have to have some factors!
>
> Schmo: Yes, they get rarer and rarer, but if you go far enough, you'll always find another.
>
> Joe: Well, have you checked through infinity to see there are always more primes?

Schmo: Well, have you checked up to infinity to see there aren't any more primes?

I imagine a lot of people thought this was a question human beings could never answer, because we can't "check up through infinity." But around 300 BCE Euclid gave a wonderful proof that the set of primes is infinite. How did he do this? He did it by showing he could win an argument:

Euclid: Hey, Joe, I hear you think there are only a finite number of primes.

Joe: That's right.

Euclid: So then you must believe there's a highest prime.

Joe: True. I don't know what it is, but eventually there is a highest prime number, and no more after that.

Euclid: Okay, so imagine multiplying all the primes on your finite list together….

Learn the ending of Euclid's proof! Present it to someone. Did you convince them?

6) Let's say we have an infinite series of one halves:

$$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots..$$

Do you agree that that series "goes to infinity"? In other words, can we make the sum as big as we want by adding more halves? But how about this infinite series? What does it add up to?

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\ldots..$$

And this one is even trickier. It's called the harmonic series, if you want to do research on it. It has a very interesting history. Will it go to infinity?

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots\ldots.$$

7) There are an infinite number of integers, but on any finite portion of the number line, there are only a finite number of integers. For example, how many integers are between 1 and 10? Fractions are different. How many fractions are between 0 and 1? Can you prove there are an infinite number? So for each integer, there are an infinite number of fractions between it and the next integer. Cool! Now prove that the number of integers and the number of fractions are equal! But wait! Are all infinities equal? What if you compare the number of integers and the number of *real numbers*?

∞) Have an infinite amount of fun!