PFS 22-6 1/5/09 ____________________

Adventures in Number Theory

Part I. Sums of Squares

1) The first few square numbers are 1, 4, 9, 16, 25.... Every number can be written as the sum of no more than four square numbers. For example:

1 = 1
2 = 1 + 1
3 = 1 + 1 + 1
4 = 4
5 = 4 + 1
6 = 4 + 1 + 1

Go as far as you can writing every whole number as the sum of at most four square numbers. Write down any patterns you notice.

2) One way to go further with this is to see what you can notice about the numbers that can be written as the sum of two squares. (For this project it's helpful to include zero as a square number. So 4 = 0 + 4, and it's the sum of two squares.)

3) Did you notice that some square numbers can be written as the sum of two other square numbers? For example, 25 = 16 + 9. Whole number solutions to $a^2 + b^2 = c^2$ are called Pythagorean triples, so 3, 4, 5 is the smallest Pythagorean triple. Can you find more Pythagorean triples? How many are there? Is there a way to find "all" of them?

4) Can you learn something of the history of the sum of squares problem? It involves many centuries and many cultures!

Part II. Factorials and Primes

Here's a chance to practice your paper-and-pencil multiplication and long division, and to make a great discovery! Don't use "scrap paper"! Do all your work neatly, save it, and show it to your POW and math teachers.

Factorials are numbers like 4 x 3 x 2 x 1. That would be called "4!" (pronounced "four factorial") and it equals 24. Factorials are used to count permutations. For example, there are 4! ways the letters ABCD can be arranged in a "word."

Primes are numbers with exactly two factors. The first few primes are 2, 3, 5, 7, 11, 13....

5) Find as many factorials as you can using paper and pencil. Try to get to 16! at least. (Happy multiplying.) Then divide each n! by (n + 1) and notice the remainder. For example, you would divide 24 (which is 4!) by 5. What patterns do you notice? Is something special happening when you divide by a prime number?

6) Here's a related adventure: We can define "primorials" as the product of the first n primes. So four primorial would be 2 x 3 x 5 x 7 = 210. Find the first few primorials. For each primorial, add one, and then test if that number is prime. (For example, is 211 prime?) What do you discover?

7) Euclid used the ideas in #6 for his amazing proof--more than two thousand years ago--that the set of prime numbers is infinite. Find out about that proof. See if you can explain it to someone!

Part III. Especially for Eighth Graders

In the bad old days, before calculators or computers, the way to test if a number was prime would be to divide by every prime number up to the square root of the number you're testing. For a big number, that's a lot of divisions! Pierre Fermat used a method based on an algebraic identity you have learned:

$$a^2 - b^2 = (a-b)(a+b)$$

If a and b are whole numbers, and a is at least two bigger than b, then this means $a^2 - b^2$ is a composite number, since it has two factors other than one and itself. Well, if

$a^2 - b^2 = N$, then $a^2 - N = b^2$

So Fermat looked at the square numbers bigger than the N he was testing and subtracted N from them. If the difference was a square number, Bingo! he had a factorization! Here's a small example. Let's test if N = 91 is a prime. The smallest square number bigger than 91 is 100. We'll start with that and put our work in a table:

$10^2 = 100$ 100 - 91 = 9 Yay! 9 is a square number! It worked on the first try!

So $91 = 100 - 9 = 10^2 - 3^2 = (10 - 3)(10 + 3) = 7 \times 13$. So 91 is composite with factors 7 and 13.

8) Test 3,233 to see if it's prime. If it's composite, what are its factors? If you used trial division, you'd have to do a lot of long divisions. Try Fermat's method instead! It might take just a few minutes! Next try one a little harder: 8,023.

9) There are other short-cuts you can add to Fermat's method. Try to learn them too.

10) Have fun!